3 Projective embedding of Riemann surfaces

3.1 Projective space

The n-dimensional complex projective space is defined by $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1}\backslash\{0\})/\sim$ with the equivalence relation $z \sim w$ in $\mathbb{C}^{n+1}\backslash\{0\} :\Leftrightarrow w = \lambda z$ for some $\lambda \in \mathbb{C}^\times$. The projective space is equipped with quotient topology making it second countable and Hausdorff. There is so-called homogeneous coordinate given by the projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$, $(z_0, z_1, \ldots, z_n) = z \mapsto [z] = [z_0 : z_1 : \cdots : z_n]$. In terms of homogeneous coordinate, one can find a standard chart determined by homeomorphisms $\varphi_i : U_i := \{[z_0 : \cdots z_n] \mid z_i \neq 0\} \to \mathbb{C}^n$, $[z_0 : \cdots : z_n] \mapsto (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i})$ for all $i = 0, \ldots, n$. The transition functions of charts are given by

$$g_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$
$$z = (z_1, \ldots, z_n) \mapsto g_{ij}(z) = \begin{cases} z/z_{i+1}^2 & i < j \\ z/z_i^2 & i > j \end{cases}$$

which are holomorphic maps on open subsets of $\mathbb{C}^n$. This demonstrates that the complex projective space is indeed a n-dimensional complex manifold.

From algebraic geometry point of view, $\mathbb{P}^n(\mathbb{C})$ is a variety equipped with Zariski topology. Let A be a generic ring, $A[z_0, \ldots, z_n]$ is a polynomial ring which is naturally a graded ring, its projective spectrum of the polynomial ring $\mathbb{P}^n_A := \operatorname{Proj} A[z_0, \ldots, z_n]$ turns out to be a scheme. When $A = \Bbbk$ (e.g. $\mathbb{C}$ for our case) is a algebraically closed field, the subspace of closed point in scheme $\mathbb{P}^n_\Bbbk$ is homeomorphic to variety $\mathbb{P}^n(\Bbbk)$. In general, over algebraically closed field $\Bbbk$, there is a fully faithful functor from the category of varieties to the category of schemes and for any variety its topological space (w.r.t.Zariski topology) is homeomorphic to the set of closed point in the space of the corresponding scheme under the functor, its sheaf of regular functions is obtained by restricting the structure sheaf of the scheme via this homeomorphism. Cf. [Hart] Ch.II §2 for full details.

3.2 Line bundles on projective spaces

Recall twisted sheaf on $\mathbb{P}^1$.

Definition 3.2.1. The twisted sheaf $\mathscr{O}(1)$ on $\mathbb{P}^n$ is the sheaf of holomorphic sections of the line bundle $L \to X$ defined w.r.t. standard charts by the cocycle $g = (g_{ij} := z_j/z_i) \in Z^1(\mathcal{U}, \mathscr{O}^\times)$.

The sections of $\mathscr{O}(1)$ are linear polynomials, i.e. $\mathscr{O}(1)(\mathbb{P}^n) = H^0(\mathbb{P}^n, \mathscr{O}(1)) \subset \mathbb{C}[z_0, \ldots, z_n]$.

3.3 Projective map

An invertible sheaf $\mathscr{L}$ on a Riemann surface X is said to be globally generated (generated by global sections) if it has no base point (a point $x \in X$ s.t. for all sections $s \in \mathscr{L}(X)$ the germ at x satisfies $s_x \in \mathfrak{m}_x\mathscr{L}$ i.e. $s(x) = 0$), namely for any point $x \in X$ there exits a section $s \in \mathscr{L}(X)$ s.t. the germ $s_x \in \mathscr{L}_x$ generates the stalk $\mathscr{L}_x$ as a $\mathscr{O}_{X,x}$-module. An invertible sheaf $\mathscr{L}$ on X is globally generated iff. for any $x \in X$ the canonical map $H^0(X, \mathscr{L}) = \mathscr{L}(X) \to \mathscr{L}_x$, $s \mapsto s_x$ is surjective.

Consider a globally generated invertible sheaf $\mathscr{L}$ on a compact Riemann surface X, By choosing a basis $(s_i)_{i=0,\dots n} \in H^0(X, \mathscr{L})$ (it is a finite dimensional topological vector space), we can define a map

$$\begin{aligned}\Phi : X &\to \mathbb{P}^n \\ x &\mapsto [s_0(x) : \dots : s_n(x)].\end{aligned}$$

Proposition 3.3.1. The map Φ is well-defined and induces an isomorphism $\Phi^*\mathscr{O}_{\mathbb{P}^n}(1) \cong \mathscr{L}$.

Proof. Pick a point $x \in X$ and consider Φ defined in a suitable open neighborhood U of x, on which we may identify $\mathscr{L}$ with $\mathscr{O}_X$. Then sections from $H^0(U, \mathscr{L})$ are holomorphic functions. Since $\mathscr{L}$ is globally geenrated, there is $s_j(x) \neq 0$ for certain $j \in \{0, \dots, n\}$. Hence the point $(s_0(x), \dots, s_n(x)) \in \mathbb{P}^n$ is well-defined and indepedent of the choice of the chart and $\mathscr{L}|_U \cong \mathscr{O}_X|_U$. Obviously Φ is holomorphic on U. For each $i \in \{0, \dots, n\}$, define $X_i := \Phi^{-1}(U_i) = \{x \in X : s_{i,x} \neq \mathfrak{m}_x\mathscr{L}_x\}$ and they form an open cover $(X_i)_{i=0,\dots,n}$ of X due to $\mathscr{L}$ is globally generated. The homomorphism $\mathscr{O}_{\mathbb{P}^n}(1)(U_i) \to (\Phi_*\mathscr{L})(U_i) = \mathscr{L}(X_i)$ induced from $z_i \mapsto s_i$ is an isomorphism. Consequently, we further obtain an isomorphism of sheaves of $\mathscr{O}_{\mathbb{P}^n}$-module $\mathscr{O}_{\mathbb{P}^n}(1) \xrightarrow{\cong} \Phi_*\mathscr{L}$. Using adjointness of direct image (as a covariant functor with inverse image as adjoint) $\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^n}}(\Phi^*\mathscr{O}_{\mathbb{P}^n}(1), \mathscr{L}) \cong \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^n}}(\mathscr{O}_{\mathbb{P}^n}(1), \Phi_*\mathscr{L})$, there is an isomorphism $\Phi^*\mathscr{O}_{\mathbb{P}^n}(1) \xrightarrow{\cong} \mathscr{L}$. □

Remark 3.3.2. The above definition of the map Φ depends on the choice of basis of $H^0(X, \mathscr{L})$. Grothendieck has proposed an intrinsic definition by the dual construction: $\Phi : X \to \mathbb{P}(H^0(X, \mathscr{L})^*)$, $x \mapsto \lambda_x$ with $\lambda_x : H^0(X, \mathscr{L}) \to \mathscr{L}_x/\mathfrak{m}_x\mathscr{L}_x \cong \mathbb{C}$, $s \mapsto [s_x]$. The value of $s(x) \in \mathbb{C}$ depends on the choice of the isomorphism between $\mathscr{L}$ and $\mathscr{O}$ in a neighborhood of x. Nevertheless the class of λ_x is independent of the choice. [Weh]

Next we will show the geometric criterion when the map induced by a globally generated invertible sheaf would be a closed embedding.

Theorem 3.3.3. Consider a globally generated invertible sheaf $\mathscr{L}$ on a compact Riemann surface X. Then the induced map $\Phi : X \to \mathbb{P}^n$ is a closed embedding iff. $\mathscr{L}$ satisfies the following properties

i) separating points: For any two distinct points $x, x' \in X$ there exists a section $s \in H^0(X, \mathscr{L})$ with $s(x) \neq 0$ but $s(x') = 0$ or the other way around.

ii) separating tangent vectors: For all $x \in X$ the map

$$\begin{aligned}d'_x : \{s \in H^0(X, \mathscr{L}) : s_x \in \mathfrak{m}_{X,x}\mathscr{L}_x\} &\to \mathfrak{m}_{X,x}\mathscr{L}_x/\mathfrak{m}^2_{X,x}\mathscr{L}_x \\ s &\mapsto [s_x]\end{aligned}$$

is surjective.

Remark 3.3.4. First, note the $\mathscr{O}_{X,x}$-modules isomorphisms

$$\mathfrak{m}_{X,x}\mathscr{L}_x/\mathfrak{m}^2_{X,x}\mathscr{L}_x \cong (\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}) \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x \cong \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x.$$

The map

$$d'_x : \{s \in H^0(X, \mathscr{L}) : s_x \in \mathfrak{m}_x\mathscr{L}_x\} \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$

is induced by the total differential $d_x = d'_x + d''_x$ acting on the holomorphic sections: a holomorphic sections $s \in H^0(X, \mathscr{L})$ with $s_x \in \mathfrak{m}_x\mathscr{L}_x$ can be written locally (in a suitable neighborhood U of x) as $s = f \cdot s'$ with holomorphic function $f \in \mathscr{O}_X(U)$ satisfying $f_x \in \mathfrak{m}_{X,x}$ and a holomorphic section $s' \in \mathscr{L}(U)$. Then $d'_x s = d'_x f \otimes s' \in \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$. This map is well-defined: if s can also be written locally as $g \cdot s''$ with $\mathrm{ord}(g; x) \geq \mathrm{ord}(f; x)$, then $g = h \cdot f$ with $h \in \mathscr{O}_X(U)$ and $h_x \in \mathfrak{m}^2_{X,x}$. Hence $d'_x(g \otimes s'') = d'_x(h \cdot f) \otimes s_2 = (h \cdot d'_x f) \otimes s'' = d'_x f \otimes h \cdot s'' = d'_x f \otimes s'$.

Set $p = \Phi(x) \in \mathbb{P}^n$ and consider on $\mathbb{P}^n$ a similar map

$$d'_p : \{\sigma \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) : \sigma_x \in \mathfrak{m}_{\mathbb{P}^n,p}\mathscr{O}_{\mathbb{P}^n}(1)_p\} \to \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p,$$

which is surjective. The pullback of sections

$$\begin{aligned} \Phi^* : H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) &\to H^0(X, \Phi^*\mathscr{O}_{\mathbb{P}^n}(1)) = H^0(X, \mathscr{L}) \\ \sigma &\mapsto s = \sigma \circ \Phi \end{aligned}$$

is surjective by definition. Consider $\mathscr{O}_{\mathbb{P}^n,p}$ and $\mathscr{O}_{X,x}$ as two base rings, the pullback of holomorphic functions induces surjective ring morphism

$$\begin{aligned} \mathscr{O}_{\mathbb{P}^n,p} &\to \mathscr{O}_{X,x} \\ f &\mapsto f \circ \Phi \end{aligned}$$

making $\mathscr{O}_{X,x}$ as a $\mathscr{O}_{\mathbb{P}^n,p}$-module. Now $\Omega^1_{\mathbb{P}^n,p}$ and $\mathscr{O}_{\mathbb{P}^n}(1)_p$ can be viewed as $\mathscr{O}_{\mathbb{P}^n,p}$-modules, while $\Omega^1_{X,x}$ and $\mathscr{L}_x = \mathscr{O}_{\mathbb{P}^n}(1)_p \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{X,x}$ can be viewed as $\mathscr{O}_{X,x}$-modules. The pullback of differential forms gives a $\mathscr{O}_{X,x}$-morphism

$$\Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{X,x} \to \Omega^1_{X,x}$$

and the composition

$$\Omega^1_{\mathbb{P}^n,p} \to \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{X,x} \to \Omega^1_{X,x}$$

is a $\mathscr{O}_{\mathbb{P}^n,p}$-morphism. Tensoring by $\mathscr{O}_{\mathbb{P}^n,p}$-module $\mathscr{O}_{\mathbb{P}^n}(1)_p$ gives a $\mathscr{O}_{\mathbb{P}^n,p}$-morphism

$$\Phi^*_{\Omega^1} : \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p = \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p) = \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$

These maps all together give a commutative diagram:

$$\begin{array}{ccc} \{\sigma \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) : \sigma(p) = 0\} & \xrightarrow{d'_p} & \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p \\ \Phi^* \downarrow & & \downarrow \Phi^*_{\Omega^1} \\ \{s \in H^0(X, \mathscr{L}) : s(x) = 0\} & \xrightarrow{d'_x} & \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x \end{array}$$

Proof. We first assume $\Phi : X \hookrightarrow \mathbb{P}^n$ is closed embedding. For a point $x \in X$, $\Phi(x) = p \in \mathbb{P}^n$, the hyperplane $H \subset \mathbb{C}^{n+1}$ containing the fiber (complex line) $L_p := \pi^{-1}(p) \subset \mathbb{C}^{n+1}$ are 1:1 corresponding to the non-zero sections $\sigma_H \in H^0(\mathbb{P}^n, \mathscr{O}(1))$ with $\sigma_H(p) = 0$: The hyperplane is represented as $H = \ker(\lambda)$ where $\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$, $(z_0, \ldots, z_n) \mapsto \sum_{j=0}^n \lambda_j z_j$ is a non-zero map. Then the section given by $\sigma_H([z_0 : \cdots : z_n]) := \sum_{j=0}^n \lambda_j z_j$ satisfies $\sigma_H(p) = 0$. Now consider two distinct points $x = \pi(u_0) = [z_0^{(0)} : \cdots : z_n^{(0)}] \neq [z_0^{(n)} : \cdots : z_n^{(n)}] = \pi(u_n) = x'$ in X. The two vectors $u_0 = (z_0^{(0)}, \ldots, z_n^{(0)}), u_n = (z_0^{(n)}, \ldots, z_n^{(n)}) \in \mathbb{C}^{n+1}$ are $\mathbb{C}$-linear independent hence they extend to a basis $\{u_i\}_{i=0,\ldots,n}$ of $\mathbb{C}^{n+1}$. Construct the hyperplane $H := \operatorname{span}_{\mathbb{C}}\{u_i : i = 0, \ldots, n-1\} \subset \mathbb{C}^{n+1}$ containing u_0 but not u_n. Define $s := \sigma_H \circ \Phi \in H^0(X, \mathscr{L})$ and this satisfies $s(x) = 0$ and $s(x') \neq 0$. Therefore $\mathscr{L}$ separates points.

Consider a point $x \in X$ and $p = \Phi(x) \in \mathbb{P}^n$. By assumption Φ is an immersion, i.e. $\Omega^1_{X,x} \to \Omega^1_{\mathbb{P}^n,p}$ is surjective, hence the map $\Phi^*_{\Omega^1} : \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$ is subjective. Further by surjectivity of d'_p, $\Phi^*_{\Omega^1} \circ d'_p$ is thus subjective as well. The commutativity of the diagram in Remark 3.3.4 implies $d'_x : \{s \in H^0(X, \mathscr{L}) : s(x) = 0\} \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$ is surjective. Hence $\mathscr{L}$ separates tengent vectors at $x \in X$.

Conversely assume $\mathscr{L}$ separate points and tangent vectors. Separating points implies Φ is injective. The map is continous and X is compact, hence the image $\Phi(X) \subset \mathbb{P}^n$ is compact and thus closed. To show $\Phi = [s_0 : \cdots : s_n] : X \to \mathbb{P}^n$ is an immersion, consider again the commutative diagram in Remark 3.3.4. By assumption d'_x is surjective, this implies $d'_x \circ \Phi^*$ is surjective and thus $\Phi^*_{\Omega^1} : \Omega^1_{\mathbb{P}^n,p} \otimes_{\mathscr{O}_{\mathbb{P}^n,p}} \mathscr{O}_{\mathbb{P}^n}(1)_p \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$ is surjective. As a consequence $\Omega^1_{X,x} \to \Omega_{\mathbb{P}^n,p}$ is surjective for all $x \in X$, which shows that Φ is indeed an immersion. □

3.4 Ample line bundles

According to the relation between holomorphic line bundles and invertible sheaves over Riemann surface as well as the projective map discussed last section, we can make definition for the notation of ample and very ample.

Definition 3.4.1.: Consider a globally generated invertible sheaf $\mathscr{L}$ on a compact Riemann surface X. The sheaf is very ample if the induced map $\Phi : X \to \mathbb{P}^n$ is a closed embedding. The sheaf is ample if there is an $N \in \mathbb{N}$ (without 0) s.t. for all $n > N$ the power $\mathscr{L}^{\otimes n}$ is very ample.

One can compare with the corresponding concepts defined for divisors. [Hart]

Now denote the sheaf of mereomorphic sections of $\mathscr{L}$ which are multiples of the divisor $-D$ as $\mathscr{L}_D := \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}_D$.

Proposition 3.4.2. Consider an invertible sheaf $\mathscr{L}$ on a compact Riemann surface X. The following statements are equivalent:

i) The sheaf $\mathscr{L}$ is very ample.

ii) For point divisors $P, P' \in \mathrm{Div}(X)$ of two arbitrary, not necessarily distint points $x, x' \in X$, $\dim H^0(X, \mathscr{L}_{-P-P'}) = \dim H^0(X, \mathscr{L}) - 2$.

Proof. *ii*) $\Rightarrow$ *i*) For any $P, P' \in \mathrm{Div}(X)$, the dimension formula implies $H^0(X, \mathscr{L}_{-P-P'}) \subsetneq H^0(X, \mathscr{L})$ is a codim. 2 proper inclusion. This factorizes as $H^0(X, \mathscr{L}_{-P-P'}) \subsetneq H^0(X, \mathscr{L}_{-P}) \subsetneq H^0(X, \mathscr{L})$ and each proper inclusion is of codim. 1. Using this fact we observe the following:

• $\dim H^0(X, \mathscr{L}_{-P}) = \dim H^0(X, \mathscr{L}) - 1$ implies

$$\operatorname{codim} \ker \left(\substack{H^0(X,\mathscr{L}) \to \mathscr{L}_x / \mathfrak{m}_{X,x}\mathscr{L}_x \cong \mathbb{C} \\ s \mapsto [s_x]} \right) = 1.$$

Hence x is not a base point of $\mathscr{L}$, i.e. $\mathscr{L}$ is globally generated, and therefore $\Phi : X \to \mathbb{P}^n$ is well-defined.

• $H^0(X, \mathscr{L}_{-P-P'}) \subsetneq H^0(X, \mathscr{L}_{-P})$ implies that for any pair of distinct points $x, x' \in X$ there exists a section $s \in H^0(X, \mathscr{L}_{-P}) \backslash H^0(X, \mathscr{L}_{-P-P'})$, i.e. $s(x) = 0$ but $s(x') \neq 0$. Hence $\mathscr{L}$ separates points.

• For any $P \in \mathrm{Div}(X)$, inclusion $H^0(X, \mathscr{L}_{-2P}) \subsetneq H^0(X, \mathscr{L}_{-P})$ is codim. 1. Then there exists a section $s \in H^0(X, \mathscr{L}_{-P}) \backslash H^0(X, \mathscr{L}_{-2P})$. This implies the composition of canonical maps $H^0(X, \mathscr{L}_{-P}) \to \mathfrak{m}_{X,x}\mathscr{L}_x \to \mathfrak{m}_{X,x}\mathscr{L}_x / \mathfrak{m}^2_{X,x}\mathscr{L}_x \cong \mathrm{span}_{\mathbb{C}}\{[s_x]\}$ is surjective since $\dim_{\mathbb{C}}(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}) = 1$. Therefore $\{s \in H^0(X, \mathscr{L}) : s_x \in \mathfrak{m}_x \mathscr{L}_x\} \twoheadrightarrow \mathfrak{m}_{X,x}\mathscr{L}_x / \mathfrak{m}^2_{X,x}\mathscr{L}_x$, i.e. $\mathscr{L}$ separates tangent vectors.

These observations implies via Theorem 3.3.3 that Φ is a closed embedding and hence *i*) holds.

i) $\Rightarrow$ *ii*) Since by assumption $\mathscr{L}$ is very ample, then by theorem 3.3.3 $\mathscr{L}$ separates points and tangent vectors. Reversing the arguments above, separating points implies for all distinct $P, P' \in \mathrm{Div}(X)$,

$\dim H^0(X, \mathscr{L}_{-P-P'}) = \dim H^0(X, \mathscr{L}) - 2$. Separating tangent vectors implies for every $P \in \operatorname{Div}(X)$, $\dim H^0(X, \mathscr{L}_{-2P}) = \dim H^0(X, \mathscr{L}) - 2$. ☐

Finally, we come to show the most important results in this note — the embedding theorem for compact Riemann surfaces.

Theorem 3.4.3. Any compact Riemann surface X has a closed embedding into $\mathbb{P}^n$ for suitable $n \geq 1$.

Proof. We first show the existence of such an embedding: The Riemann-Roch theorem for invertible sheaf over compact Riemann surface X with genus g states $\chi(\mathscr{L}) = 1 - g + \langle c_1(\mathscr{L}), [X]\rangle$ and $\chi(\mathscr{L}_{-P-P'}) = 1-g+\langle c_1(\mathscr{L}), [X]\rangle - 2$ for any $P, P' \in \operatorname{Div}(X)$, which implies $\chi(\mathscr{L}_{-P-P'}) = \chi(\mathscr{L}) - 2$. Once we can find an invertible sheaf so that $\dim H^1(X, \mathscr{L}) = \dim H^1(X, \mathscr{L}_{-P-P'}) = 0$, then by the criteria of very ampleness of $\mathscr{L}$ there exists such an embedding. We reformulate this dimension vanishing condition by Serre duality of invertible sheaves as $\dim H^0(X, \mathscr{L}^* \otimes_{\mathscr{O}_X} \otimes \omega_X) = \dim H^0(X, \mathscr{L}^*_{-P-P'} \otimes_{\mathscr{O}_X} \omega_X) = 0$. We know from before a necessary condition for the vanishing of these dimensions is

$$\langle c_1(\mathscr{L}^* \otimes_{\mathscr{O}_X} \otimes \omega_X), [X]\rangle = -\langle c_1(\mathscr{L}), [X]\rangle + \langle c_1(\omega_X), [X]\rangle < 0$$
$$\langle c_1(\mathscr{L}^*_{-P-P'} \otimes_{\mathscr{O}_X} \omega_X), [X]\rangle = -\langle c_1(\mathscr{L}), [X]\rangle + \langle c_1(\omega_X), [X]\rangle + 2 < 0$$

and $\langle c_1(\omega_X), [X]\rangle = 2(g-1)$. Hence the claim reduces to the existence of an invertible sheaf $\mathscr{L}$ with $\langle c_1(\mathscr{L}), [X]\rangle > 2g$. For any $D \in \operatorname{Div}(X)$, $\langle c_1(\mathscr{O}_D), [X]\rangle = \deg D$. Therefore any sufficiently high multiple of a point divisor on X provides a suitable invertible sheaf $\mathscr{L}$.

Next we find such sheaf by explicit construction in various cases.

$g = 0:$ We choose a point divisor $P \in \operatorname{Div}(X)$ and set $\mathscr{L} = \mathscr{O}_P$. Then $\dim H^0(X, \mathscr{L}) = 1-0+1 = 2$, namely $H^0(X, \mathscr{L})$ spanned by two global holomorphic sections. Hence $\Phi : X \to \mathbb{P}^1$ provides a closed embedding and by the topology property it is actually an isomorphism.

$g = 1:$ We choose a point divisor $P \in \operatorname{Div}(X)$ and set $\mathscr{L} = \mathscr{O}_{3P}$. Then $\dim H^0(X, \mathscr{L}) = 1 - 1 + 3$, namely $H^0(X, \mathscr{L})$ is spanned by three global holomorphic sections. Hence $\Phi : X \to \mathbb{P}^2$ provides a closed embedding.

$g \geq 2:$ We set $\mathscr{L} = \omega_X^{\otimes 3}$, namely the sheaf of sections of the tri-canonical bundle.. Then $\langle c_1(\mathscr{L}), [X]\rangle = 3 \cdot 2(g-1) > 2g$. It follows $\dim H^0(X, \mathscr{L}) = 1 - g + 6(g-1) = 5(g-1)$. Therefore $\Phi : X \to \mathbb{P}^{5(g-1)-1}$ provides a closed embedding. ☐

Remark 3.4.4

• The theorem shows that for genus $g \geq 2$ the tri-canonical bundle provides a projective embedding of X. For $g = 1$ the canonical bundle is trivial, i.e. $\omega_X \cong \mathscr{O}_X$, hence for each power of the canonical bundle the map Φ maps X to a point. For $g = 0$ no positive power of the canonical bundle has a holomorphic section.

• More generally, one can prove thatthere always exist closed embeddings into $\mathbb{P}^3$, see [Hart] Chapter IV, Cor. 3.6.

• The proof of the theorem shows that the only compact Riemann surface with genus zero is the projective space $\mathbb{P}^1$. An analogous statement does not hold for higher genus: the moduli space of compact Riemann surfaces has the dimension

$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 0 & g = 0 \\ 1 & g = 1 \\ 3g-3 & g \geq 2 \end{cases}$$